CONGRUENCE PERMUTABILITY IN QUASIVARIETIES

LUCA CARAI, MIRIAM KURTZHALS, AND TOMMASO MORASCHINI

Abstract. It is shown that a natural notion of congruence permutability for quasivarieties already implies "being a variety". The result follows immediately from [3] and the sole aim of this note is to state it explicitly, together with a telegraphic proof.

We denote the class operators of closure under isomorphic copies, subalgebras, homomorphic images, direct products, and ultraproducts by $\mathbb{I}, \mathbb{S}, \mathbb{H}, \mathbb{P}$, and $\mathbb{P}_u$, respectively. A class of algebras is said to be:

(i) a *variety* when it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$;
(ii) a *quasivariety* when it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_u$.

While every variety is a quasivariety, the converse is not true in general. We call *proper* the quasivarieties that are not varieties. Examples of a proper quasivariety include the class of cancellative commutative monoids.

As quasivarieties need not be closed under $\mathbb{H}$, the following concept is often useful. Let K be a quasivariety and $\boldsymbol{A}$ an algebra (not necessarily in K). A congruence θ of $\boldsymbol{A}$ is said to be a K-*congruence* when $\boldsymbol{A}/\theta \in \mathsf{K}$. When ordered under the inclusion relation, the set of K-congruences of $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ in which meets are intersections (see, e.g., [4, Prop. 1.4.7 & Cor. 1.4.11]). When K is the quasivariety of all algebras of a given type, $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ coincides with the congruence lattice $\mathsf{Con}(\boldsymbol{A})$ of $\boldsymbol{A}$. Given a quasivariety K and an algebra $\boldsymbol{A}$, we denote by $\wedge, \vee$ and $\wedge^{\mathsf{K}}, \vee^{\mathsf{K}}$ the meet and join operations in $\mathsf{Con}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ respectively. Moreover, we will rely on the following observation, which is an immediate consequence of [1, Lem. 4.2]: for every $a, b \in A$ and $X \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$,

$$\langle a, b\rangle \in \bigvee^{\mathsf{K}} X \iff \text{there exists a finite } Y \subseteq X \text{ such that } \langle a, b\rangle \in \bigvee^{\mathsf{K}} Y. \tag{1}$$

The following is a straightforward corollary of the Homomorphism Theorem (see, e.g., [2, Thm. II.6.12]).

Proposition 1. *A quasivariety* K *is a variety if and only if* $\mathsf{Con}(\boldsymbol{A}) = \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *for every* $\boldsymbol{A} \in \mathsf{K}$.

Given two binary relations R_1 and R_2 on a set A, we let

$$R_1 \circ R_2 = \{\langle a, b\rangle \in A \times A : \text{there exists } c \in A \text{ s.t. } \langle a, c\rangle \in R_1 \text{ and } \langle c, b\rangle \in R_2\}.$$

A variety K is said to be *congruence permutable* when for every $\boldsymbol{A} \in \mathsf{K}$ and $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$ we have $\theta \circ \phi = \phi \circ \theta$. Equivalently, K is congruence permutable if and only if $\theta \vee \phi = \theta \circ \phi$ for every $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$ (see, e.g., [2, Thm. 5.9]). We will prove that the analogous version of the latter property for quasivarieties implies "being a variety". This observation is a direct consequence of the results in [3], as shown in the next proof.

Theorem 2. *Let* K *be a quasivariety such that* $\theta \vee^{\mathsf{K}} \phi = \theta \circ \phi$ *for every* $\boldsymbol{A} \in \mathsf{K}$ *and* $\theta, \phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. *Then* K *is a variety.*

Proof. In view of [3], it suffices to show that $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a complete sublattice of $\mathsf{Con}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{K}$. To this end, consider $\boldsymbol{A} \in \mathsf{K}$. As meets are intersections in both $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and $\mathsf{Con}(\boldsymbol{A})$ and $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \subseteq \mathsf{Con}(\boldsymbol{A})$, it suffices to show that

$$\bigvee^{\mathsf{K}} X \subseteq \bigvee X \text{ for every } X \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}). \tag{2}$$

We begin with the following observation.

Claim 3. *For every* $\theta, \phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *we have* $\theta \vee^{\mathsf{K}} \phi \subseteq \theta \vee \phi$.

Proof of the Claim. Since $\theta \vee \phi$ contains $\theta \circ \phi$ (see, e.g., [2, Thms. I.4.7 & II.5.3]) and the assumptions ensure that $\theta \circ \phi = \theta \vee^{\mathsf{K}} \phi$, we conclude that $\theta \vee^{\mathsf{K}} \phi \subseteq \theta \vee \phi$. ⊠

To prove (2), consider $\langle a, b \rangle \in \bigvee^{\mathsf{K}} X$. By (1) there exists a finite $Y \subseteq X$ such that

$$\langle a, b \rangle \in \bigvee^{\mathsf{K}} Y. \tag{3}$$

First, suppose that Y is nonempty. By applying in sequence (3), Claim 3, and $Y \subseteq X$, we obtain

$$\langle a, b \rangle \in \bigvee^{\mathsf{K}} Y \subseteq \bigvee Y \subseteq \bigvee X$$

and we are done. Next, suppose that Y is empty. In this case, $\bigvee^{\mathsf{K}} Y$ is the minimum of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. As $\boldsymbol{A} \in \mathsf{K}$ by assumption, this minimum is the identity relation on A. Together with (3), this yields $a = b$. Consequently, $\langle a, b \rangle$ belongs to every congruence of $\boldsymbol{A}$ and, in particular, to $\bigvee X$. ⊠

We remark that Theorem 2 does not imply that every subquasivariety of a congruence permutable variety V is also a variety (counterexamples are well known and easy to find, e.g., in the case where V is the variety of all Heyting algebras).

References

[1] W. J. Blok and J. G. Raftery. Ideals in quasivarieties of algebras. In *Models, algebras, and proofs (Bogotá, 1995)*, volume 203 of *Lecture Notes in Pure and Appl. Math.*, pages 167–186. Dekker, New York, 1999.

[2] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra.* 2012. The millennium edition, available online.

[3] M. Ćirić and S. Bogdanović. Posets of $\mathfrak{C}$-congruences. *Algebra Universalis*, 36(3):423–424, 1996.

[4] V. A. Gorbunov. *Algebraic theory of quasivarieties.* Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998. Translated from the Russian.

Luca Carai: Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy

Email address: luca.carai.uni@gmail.com

Miriam Kurtzhals and Tommaso Moraschini: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain

Email address: mkurtzku7@alumnes.ub.edu and tommaso.moraschini@ub.edu